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THEORY OF THE CURRENT–VOLTAGE CHARACTERISTICS OF SNS JUNCTIONS AND OTHER SUPERCONDUCTING WEAK LINKS

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A theory of the current–voltage characteristics of superconducting-normal-superconducting junctions is proposed. The theory is based on a simple model current–density equation. The form of the current–voltage characteristic is found to depend critically on whether the electrical source should be treated as a current source or a voltage source at high frequencies. A picture is given of the dynamical state of a junction at finite voltage, and all the important differences between the characteristics observed for SNS junctions and tunnel junctions are explained at least qualitatively. In particular, the theory covers the cases of a junction wide enough to be limited by its own field and the step structure known to be induced by an applied h.f. field. For wide junctions the predicted behaviour is an exactly calculable quantized flux flow in one dimension. It is the phase-locking of this flux flow by the applied h.f. field which leads to the step structure. A detailed calculation of the rounding of the steps by noise is given; the steps are predicted to be exceedingly steep. The theory is compared with the theories of other superconducting weak links.

1. INTRODUCTION

Two superconductors which are very loosely coupled together so that only a small supercurrent may pass between them are said to form a weak link. Numerous types of weak link have been studied since the system was first discussed theoretically by Josephson (1962) and it is convenient for the present purpose to divide them into three broad classes. In the first belong the oxide tunnel junction first studied experimentally by Anderson & Rowell (1963) and the superconductor-normal-superconductor (SNS) junction (Clarke 1969). These junctions have barriers which are not inherently superconducting and there are theoretical reasons for believing that at zero voltage the supercurrent density (j_S) flowing through them is described by the relation (Josephson 1962, 1964, 1965; de Gennes 1964):

$$j_S = j_J \sin \phi\ (\boldsymbol{r}, t), \tag{1}$$

where j_J is the Josephson critical density and $\phi(\boldsymbol{r}, t)$ is the difference between the phases of the two superconductors. Secondly, we have junctions in which the weak link is a superconductor which either is of very small cross-section such as the Dayem bridge (Anderson & Dayem 1964; Dayem & Wiegand 1967), or in which the order parameter in the link is depressed by, for example, the proximity effect of a superimposed normal metal (Notarys & Mercereau 1969). Finally we have a large number of junctions which probably lie between the two extremes. These include point contacts (Zimmerman & Silver 1966), crossed superconducting wires (Zimmerman & Silver 1966), and solder-niobium devices (Clarke 1966).

Experimentally, these systems have various properties in common. They all sustain a d.c.

supercurrent up to a certain critical value above which there occurs a transition to a finite voltage régime. In this state, the supercurrent has an a.c. component at the frequency ω given by (Josephson 1962, 1964, 1965)

$$\hbar\omega = 2\Delta\mu, \tag{2}$$

where $\Delta\mu$ is the total electrochemical potential difference across the junction. When the junctions are excited by electromagnetic radiation at frequency ω_0, steps or spikes appear at the multiple voltages $n\hbar\omega_0/2e$, corresponding to quantum processes in which n photons are absorbed or emitted and one condensed pair crosses the junction. Structure at the submultiple voltages $n\hbar\omega_0/2em$ corresponding to processes involving m pairs is also commonly observed. Junctions of all types, provided they are sufficiently large, can contain quantized flux lines, the motion of which modifies the shape of the current–voltage characteristics.

The purpose of this paper is to present a simple model of the SNS junction capable of describing its behaviour under a variety of conditions. It seems likely that some of the ideas may be extended, at least qualitatively, to other types of junction and we shall attempt to indicate where this is so.

In §2 we described the experimental i–V characteristics of the different types of junction. In §3 we present our model for the SNS junctions, and discuss the important question of the magnitudes of the internal impedances of the a.c. and d.c. sources. Section 4 is concerned with the 'small' junction, that is, one in which the supercurrent flows uniformly across the barrier, in the case when the current flows from a current source. We compute the form of the i–V characteristic both with and without external radiation. An important feature of our argument is the vital role played by the source impedances in determining the characteristic SNS behaviour. In particular we show that the structure at the submultiple voltages can be explained without modifying Josephson's supercurrent relation (1) and without appealing to flux flow, which is not a meaningful notion in small junctions. The results are extended to the case of finite source impedance in an appendix. In §5 we describe the 'large' junction in which the current flow tends to be confined to the edges of the barrier. The static vortex structure below the critical current and the modification of the i–V characteristic when the vortices are in motion are described. The induced-step structure is also discussed. Section 6 is concerned with the effect of noise on the steepness of the induced steps. We finally summarize our conclusions in §7.

This paper may be compared with the earlier paper by Werthamer & Shapiro (1967) which presented analogue computer calculations on a Josephson junction. Our model is rather different from theirs and is concerned with different aspects of the problem.

A preliminary account of our work was given at the Conference on the Science of Superconductivity held at Stanford, California, in August 1969 and will appear in the conference proceedings.

2. Experimental properties of weak links

(i) *Definitions of 'large' and 'small' junctions*

The penetration depth of a one-dimensional Josephson junction, that is, the characteristic distance from the edge of the junction over which supercurrents tend to flow, is given by (Ferrell & Prange 1963; Josephson 1964, 1965).

$$\lambda_J = (\hbar/2e\mu_0 j_J d)^{\frac{1}{2}}, \tag{3}$$

where d is the magnetic thickness, equal to the sum of the barrier thickness and the penetration depths of the two superconductors. For an SNS junction, the penetration depths may be slightly greater than their normal bulk values because of the depression of the order parameter in the superconductors near the contact with the normal metal (de Gennes 1966). If the junction has width W, we call the junction 'small' when $\lambda_J/W \gg 1$, so that the supercurrent flow is uniform, and 'large' when $\lambda_J/W \ll 1$, so that the supercurrents are screened out from the interior of the junction.

(ii) *Tunnel junctions*

The ideal low-frequency i–V characteristic of a tunnel junction seems to consist of an ordinary tunnelling characteristic with a supercurrent 'spike' at zero voltage. Thus when the critical current is exceeded there is usually a discontinuity in voltage as the junction makes a transition to a stable region. As the current is reduced again, the transition back to the zero voltage régime often occurs at a much lower current (Anderson & Rowell 1963). As one would expect the exact form of the behaviour observed depends critically upon the d.c. impedance of the current source.

When microwave excitation is present additional structure is observed on the characteristic at the multiple voltages $n\hbar\omega/2e$ but not at submultiple voltages. It seems that the structure consists fundamentally of 'spikes' which are observed as such when the d.c. current source has low impedance (Shapiro 1963) and as steps when it has high impedance (Parker, Langenberg, Denenstein & Taylor 1969 for example). For all tunnel junctions, the microwave source was probably effectively a voltage source (see §4).

(iii) *SNS junctions*

The critical current of an SNS junction increases very rapidly as the temperature is lowered. Thus a given junction may be very 'small' at a relatively high temperature and very 'large' at a lower temperature through the dependence of λ_J upon j_J given in (3). In a small junction, the dependence of critical current upon applied magnetic field is a Fraunhofer-like pattern; the relation in a large junction is modified. These results strongly imply that the relation $j = j_J \sin \phi$ is applicable to the SNS junction (Clarke 1969).

The most noticeable feature of the i–V curves of SNS junctions is that hysteresis and negative resistance regions are never observed either in the excited or in the unexcited characteristics. As the current in a small junction is increased through the critical value, the i–V characteristic tends smoothly to a straight line passing through the origin. In a large junction, the asymptote of the characteristic is parallel to this line but cuts the current axis at a value often much nearer the critical current than zero (see figure 5). We shall give an explanation of this result in §5. A more detailed account of these properties will be published shortly elsewhere.

(iv) *Dayem bridge*

The Dayem bridge consists of a superconductor of very small cross-section between two bulk superconductors (Anderson & Dayem 1964; Dayem & Wiegand 1967). With rising current the i–V characteristic tends rather rapidly to a linear characteristic which again extrapolates to a finite current at zero voltage. This is usually described as a flux-flow régime. Microwaves usually produce abundant harmonic and subharmonic structure. Hysteresis and negative resistance regions are not usually observed. It appears that in all of the studies both the a.c. and the d.c. sources had high impedance.

(v) *Intermediate systems*

Point contacts, etc., have properties which vary between those of the tunnel junction and the Dayem bridge. Their exact structure is not well understood.

(vi) *Sharpness of induced steps or spikes*

Noise may round the microwave-induced structure of all types of junction and make the differential resistance of the steps or spikes non-zero. Parker *et al.* (1969)reported that the steps on tunnel junctions with resistances above $0.1\ \Omega$ were noticeably non-vertical. The effect was even more pronounced in the case of point contacts. On the other hand, one of us (Clarke 1968), found that the steps on the SNS junctions were exceedingly sharp, having a differential resistance of less than $2\times 10^{-14}\ \Omega$.

Noise may either be generated in the junction itself (for example, Johnson noise) or picked up from external sources. Finnegan *et al.* (1969) have reported that microwave-induced steps on tunnel junctions or point contacts which have a finite slope when the cryostat is in the open laboratory sharpen up considerably when the cryostat is moved into a screened room. Thus it seems probable that external noise dominates the internal noise for these junctions unless special precautions are taken. In the case of the SNS junctions, their very low internal impedance makes the r.f. coupling to the external world extremely weak so that the noise signal picked up is normally insignificant.

3. Model for an SNS junction

(i) *Junction equations*

So far as we know no microscopic theory of the current flow in an SNS junction at finite voltage exists. At zero voltage, as we indicated above, there is good experimental evidence for the Josephson supercurrent relation (1), and de Gennes (1964) actually predicted a relation of this type for thick SNS junctions in the limit where the pair potential Δ is small. At finite voltage we expect excitations to be important in carrying current across the normal part of the junction. As the charges cross the junction supercurrent must be converted to normal current and then back again. How and where this occurs and how this process affects equation (1) for the supercurrent are important questions which we do not attempt to answer. We simply assume that for sufficiently small voltages we may write

$$j = j_S + j_N = j_J \sin\phi(\boldsymbol{r}, t) + \sigma V, \tag{4}$$

where σ is an effective conductance per unit area. Following Josephson, we are using a gauge in which the vector potential is parallel to the junction surface; div $\boldsymbol{A}$ is non-zero in general in this gauge. We are assuming that the presence of a small voltage does not appreciably modify the supercurrent term, and that the flow of excitations can be described by an approximately constant conductance. These assumptions may well be shown to be rather crude approximations. They seem more likely to be valid over a useful range of voltage in sandwiches where the normal metal has a rather small interaction parameter, so that the excitations in it are virtually normal and not likely to be much affected by the supercurrent. In that case σ might be expected to be close to the value calculated for the known thickness of bulk normal metal; the measured resistances of Pb–Cu–Pb junctions at large currents are in reasonable agreement with this idea (Clarke 1969).

Our assumptions are also in line with Josephson's microscopic calculation for tunnel junctions, admittedly a rather different case, which gave the result

$$j = j_J(V)\sin\phi + [\sigma_S(V)\cos\phi + \sigma_N(V)]V, \tag{5}$$

where $j_J(V)$ is a comparatively slow function of V, $\sigma_S(V)$ is zero at $V = 0$, and $\sigma_N(V)$ is the ordinary tunnelling conductance (Josephson 1962). Our attitude to equation (4) is that it represents a simple model which contains enough of the real physics to give a rather satisfactory account of the known properties of SNS junctions.

An important question is how far equation (4) might be expected to apply to other systems. At first sight Josephson's microscopic result (5) suggests that (4) might provide a good model for tunnel junctions. The main difficulty here is the fact that the tunnelling conductance $\sigma_N(V)$ is strongly non-linear except close to the transition temperature. Since we treat σ as a constant we would expect to find only qualitative agreement in general. A second difficulty is that tunnel junctions have a large internal capacity, whereas charge storage in SNS junctions is unimportant at frequencies below the plasma frequency. For small tunnel junctions the capacity can be treated as part of the external circuit (see next section), but for large junctions it makes transmission-line modes possible (Josephson 1966) which have no analogue for SNS junctions. The inclusion of capacitance effects in our theory of large junctions is not a simple matter, and we have not attempted to cover it in this paper. We see no reason to expect equation (4) to apply to the superconducting bridge type of weak link. The similarity between the excited i–V characteristics for superconducting bridges and those for large SNS junctions is probably explained by the fact that both cases involve flux line motion and its synchronization with the applied field (Anderson 1967). However, the flux lines in an SNS or tunnel junction which are discussed in §5 below are rather different in character from the ordinary thin-film flux lines (Tinkham 1963) which are presumably present in Dayem bridges.

In addition to the current density equation (4) we assume the usual equations for $\partial\phi/\partial t$ and $\partial\phi/\partial x$ (Josephson 1965)

$$\hbar\,\partial\phi/\partial t = 2eV, \tag{6}$$

$$\hbar\,\partial\phi/\partial x = 2edB. \tag{7}$$

Here the z axis is normal to the barrier, the magnetic field B is chosen to lie wholly parallel to the y axis and the system is invariant in the y direction. We are here restricting the treatment to a two-dimensional case which we shall refer to as the 'long junction'. This restriction is for convenience; there is no special problem in treating a more general junction if required.

(ii) *Source impedance*

In calculating the i–V characteristic for the excited junction (in which we hope to find constant voltage steps) it is important to consider carefully the source impedances for both the d.c. and the a.c. input. The traditional approach is to assume a small source impedance corresponding to a voltage source. If this has the form $V_0 + V_1\cos(\omega t)$ corresponding to a combined d.c. and a.c. input, then for a small junction we have, using (6)

$$\begin{aligned}\phi &= \frac{2e}{\hbar}\int[V_0 + V_1\cos(\omega t)]\,dt\\ &= \phi_0 + \omega_0 t + (\omega_1/\omega)\sin(\omega t),\end{aligned} \tag{8}$$

where ω_0 and ω_1 are $2eV_0/\hbar$ and $2eV_1/\hbar$. For our model the current density equation (4) then gives

$$j = j_J\sin[\phi_0 + \omega_0 t + (\omega_1/\omega)\sin(\omega t)] + \sigma V. \tag{9}$$

The supercurrent term is phase modulated and has sidebands at frequencies $\omega_0 \pm n\omega$ of amplitude $A_n = j_J\ J_n(\omega_1/\omega)$, where J_n is the Bessel function of order n. In general it consists of oscillating terms, but when $\omega_0 \pm n\omega$ is zero there is a d.c. contribution whose magnitude depends on the value of ϕ_0 but lies in the range $\pm A_n$. The ideal characteristic therefore consists of the normal characteristic with a series of supercurrent 'spikes' superimposed at the multiple voltages $nV_\omega = n\hbar\omega/2e$. This corresponds to what is observed for tunnel junctions. We note in passing that the absence of structure at the submultiple voltages nV_ω/m in this case of low source impedance is connected with the form of the Josephson supercurrent relation. If the term $\sin\phi$ were replaced by any other periodic function of ϕ it is easy to see that additional 'spikes' would appear at the submultiple voltages.

Table 1. Parameters of weak links used in typical experiments

	SNS junctions	tunnel junction	Dayem bridge
critical supercurrent—approx. range observed/A ...	10^{-5} upwards	10^{-5}–10^{-1}	10^{-6} upwards
typical experiment:	(Clarke 1969)	(Anderson & Rowell 1963)	(Dayem & Wiegand 1967)
critical supercurrent/A	10^{-2}	10^{-3}	10^{-3}
source frequency/Hz	10^{6}	10^{11}	10^{10}
voltage of fundamental step, V_ω/V	2×10^{-9}	2×10^{-4}	2×10^{-5}
internal resistance/Ω	10^{-7}	10^{-1}†	10^{-2}
reactance of supercurrent at source frequency/Ω	2×10^{-7}	2×10^{-1}	2×10^{-2}
magnetic thickness/μm	1	10^{-1}	—
Josephson penetration depth/mm	0.03	0.3	—
width/mm	0.2	0.2	0.001

† This is the tunnel resistance in the normal state. At low voltages and temperatures, the i-V curve is non-linear and the effective internal resistance can easily rise by three orders of magnitude.

In SNS junctions structure appears at both the multiple and the submultiple voltages, and it appears to consist of ideal constant voltage steps with no negative resistance regions, rather than spikes. Moreover, in large junctions there seems to be part of the d.c. supercurrent which persists up to large voltages. We believe that all these differences from the tunnel junction behaviour are due essentially to the fact that the a.c. source has a relatively high internal impedance. As we shall show, use of a current source at the outset leads to a strikingly different set of equations which do indeed predict behaviour of the sort observed. It is therefore important to examine the experimental parameters of typical weak links set out in table 1. One must remember that by varying the geometry and the temperature it is possible to make any type of weak link with such a wide variety of critical parameters that no one example can really be treated as typical. In the table we have rather arbitrarily chosen values corresponding to experiments with which we are familiar. Whether our arguments apply in other cases the reader must judge for himself. The figures for the reactance of the supercurrent are approximate and were obtained by an order of magnitude calculation in the following way. Differentiating the Josephson current relation gives

$$\partial i_S/\partial t = i_J \cos\phi\,(2eV/\hbar). \tag{10}$$

For small signals this relation is like that for an inductance of magnitude $(\hbar/2ei_J\cos\phi)$, which has a reactance of order V_ω/i_J at frequency ω. We assume that this value is representative of the effective reactance of the supercurrent under general conditions.

In considering the source impedances for the various experiments we first note that the total internal impedance of the SNS junction is very low, of the order of 10^{-7} Ω. The a.c. source at the frequency of 1 MHz was an ordinary network of wires and electronic components whose internal impedance is certainly very much greater; we can safely assume that the a.c. source was a current source. For the tunnel junction, on the other hand, the impedance to current flow across the junction is of order 0.1 Ω or more. In parallel with this impedance is the self-capitance of the junction, which is of order 10 nF, with an impedance of only 0.16 mΩ at 10^{11} Hz. So far as the real current flow across the junction is concerned, this impedance is in parallel with the source. The effective source impedance is therefore much less than the junction impedance; we have an effective a.c. voltage source. The Dayem bridge has a current flow impedance of about 10 mΩ, and negligible self-capacity. It is difficult to estimate the apparent internal impedance of the structure delivering the microwaves to the bridge, but it is unlikely to be as low as 10 mΩ; the Dayem bridge probably sees an a.c. current source. These conclusions are of course only valid for the particular cases considered. By changing the parameters, especially the frequency, they could be substantially modified for the tunnel junction and the Dayem bridge; it seems almost impossible to construct an a.c. voltage source for an SNS junction however. In the theory that follows we shall assume that we have an a.c. current source unless any other situation is specified explicitly.

4. Small junction behaviour

(i) *The basic equation*

We first consider the case of an SNS junction of dimensions considerably less than the Josephson penetration depth, so that the current density can be taken as uniform. This limit is rather hard to reach in practice except very close to T_c where the voltages and currents involved are very small, but the model shows some interesting behaviour. How far it agrees with experiment will be considered in §5. For an input current density of the form $j_0 + j_1 \sin(\omega t)$ (4) becomes

$$\begin{aligned} j_0 + j_1 \sin(\omega t) &= j_J \sin\phi + \sigma V \\ &= j_J \sin\phi + \sigma \hbar \dot{\phi}/(2e) \end{aligned} \tag{11}$$

using (6). Re-arranging in dimensionless form we have

$$\mathrm{d}\phi/\mathrm{d}u = A_0 + A_1 \sin u - A_J \sin\phi, \tag{12}$$

where u is ωt. The A variables are related to the j variables through the relation $A = j/j_\omega$ where j_ω is σV_ω, the normal current density corresponding to the voltage of the fundamental step $V_\omega = \hbar\omega/2e$. It is this differential equation for ϕ which replaces (9) when the source impedance is high.† The quantity $\mathrm{d}\phi/\mathrm{d}u$ is the instantaneous voltage in reduced units, V/V_ω. It has both oscillating and d.c. parts. If (12) can be solved to find the trajectory of the system in the ϕ, u plane, the d.c. voltage will be the long-range mean slope of this trajectory, $\overline{\mathrm{d}\phi/\mathrm{d}u}$.

(ii) *The unexcited characteristic*

In the absence of excitation ($A_1 = 0$) equation (12) has analytic solutions (McCumber 1968; Aslamazov, Larkin & Ovchinnikov 1968), some of which are illustrated in figure 1, with the corresponding i–V curve which at finite voltage has the simple form

$$j_0^2 = j_J^2 + (\sigma V)^2. \tag{13}$$

The solutions can be understood as follows. For current densities less than j_J there are time-independent solutions of the form $\sin\phi = j_0/j_J$. Since $\dot{\phi}$ is zero, these have zero voltage and represent points on the zero voltage step (case 1 in figure 1). For larger currents, ϕ increases

† A superficially similar equation obtained by Stephen is discussed in the appendix.

with time, but it does not do so uniformly as it would have done with a voltage source; the system actually spends more time in the forward supercurrent than in the backward supercurrent condition. This means that a d.c. supercurrent component persists at finite voltages, and only disappears gradually as the current rises (cases 2, 3 and 4). The a.c. supercurrent will include components not only at the Josephson frequency $2eV/\hbar$, but also at all its harmonics. This harmonic content will be strongest at low voltages.

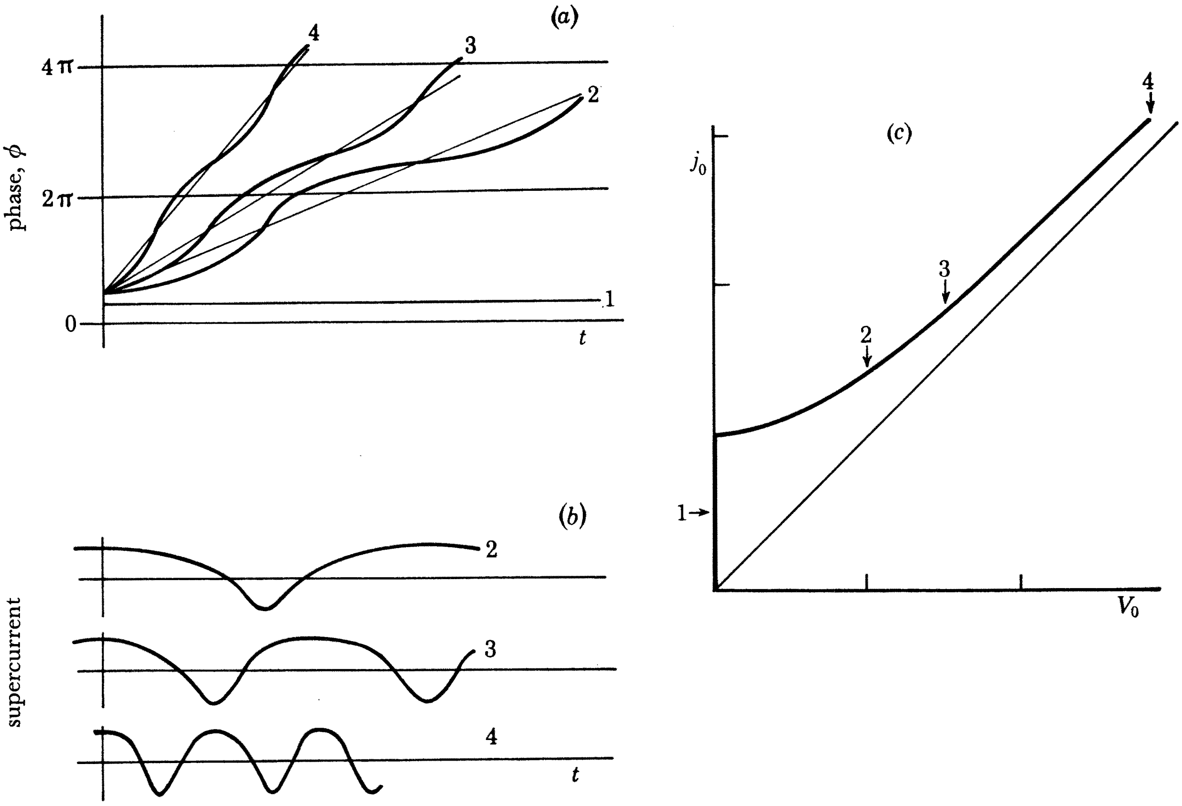

FIGURE 1. (a) Solutions of the equation $\mathrm{d}\phi/\mathrm{d}u = A_0 - A_J \sin\phi$ showing how the phase difference ϕ across the junction varies with reduced time, u, for four different values of reduced d.c. current density A_0. (b) Corresponding curves showing supercurrent across the junction as a function of time. (c) The corresponding d.c. current–voltage characteristic; the thin line on the graph represents the normal part of the current.

(iii) *The excited characteristic*

To understand the solutions of (12) with A_1 non-zero we notice that $\mathrm{d}\phi/\mathrm{d}u$ has a constant part A_0 and a part with lattice periodicity in the ϕ, u plane. We guess that an arbitrary trajectory $\phi(u)$ will sample the plane fairly uniformly so that the periodic part of $\mathrm{d}\phi/\mathrm{d}u$ will average to near zero. We therefore expect that the long-term slope $\overline{\mathrm{d}\phi/\mathrm{d}u}$ will be close to A_0. In general this is true, but when the long-term slope is close to a rational number n/m, the trajectory has a tendency to lock on to the lattice, so that the long term slope (and hence the voltage) remains locked at n/m for a finite range of values of A_0 (or d.c. current). This mechanism can produce submultiple as well as multiple voltage steps. The mechanism of the locking is most simply illustrated by considering the fundamental step for the special case $A_1 = A_J$. Schematic contours of the periodic part of $\mathrm{d}\phi/\mathrm{d}u$ are shown in figure 2. It is clear that a trajectory starting at the origin with $A_0 = 1$ will be a straight line as shown (AA). There will also be, for some value of A_0, a trajectory such as CC, with the same long-term slope as AA, but passing close to the regions of maximum and minimum slope. This trajectory will clearly oscillate in the manner shown. The fact that it does so leads to an important result: *the trajectory spends more time (extent of u) in the small-slope than the large-slope regions*. Averaging equation (12) shows that this would reduce $\overline{\mathrm{d}\phi/\mathrm{d}u}$ unless A_0 is increased to compensate this effect. So if both trajectories are to have the same mean slope (voltage) they must have different values of A_0 (current), and locking-on has occurred. In this case trajectory AA corresponds to the bottom of the current step, and trajectory CC (which passes through the strongest part of the lattice) to the top. Intermediate currents correspond to intermediate trajectories such as BB, and greater or smaller currents would give trajectories which did not lock-on with slope unity, but wandered across the ϕ, u plane with greater or smaller mean slopes.

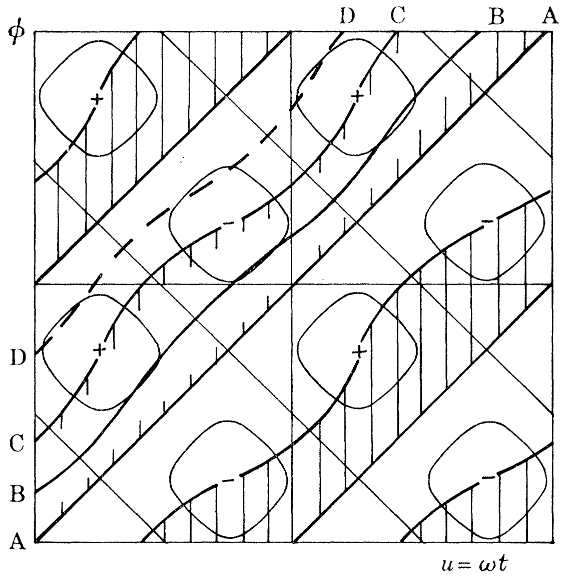

FIGURE 2. Solutions of the small junction equation $\mathrm{d}\phi/\mathrm{d}u = A_0 + A_0 \sin u - A_J \sin \phi$ showing how the phase across the junction varies with time for different values of reduced d.c. current density A_0. The trajectories shown correspond to typical periodic solutions for values of d.c. current density lying on the fundamental step of the current–voltage characteristics (heavy lines). We have chosen the special case of $J_1 = J_J$, and to help follow the trajectories we also show schematic contours of the periodic part of $\mathrm{d}\phi/\mathrm{d}u$ which is $A_1(\sin u - \sin \phi)$ (thin lines); the + and − signs indicate maxima and minima of this function. Trajectories within the shaded areas are stable. AA is the trajectory corresponding to the bottom of the step. BB is a typical trajectory for a current about half-way up the step; DD is an *unstable* trajectory corresponding to the same current. CC corresponds to the top of the step.

We have computed solutions of (12) for a wide range of values of A_1 and A_J; two of the i–V characteristics obtained are shown in figure 3. The following general rules seem to apply to the results and are at least qualitatively consistent with the known observations on SNS junctions (Clarke, to be published). Where the rule has an analytic basis we indicate the fact.

(1) It can be proved that the d.c. voltage is a monotonically increasing continuous function of the current, so that there are no negative resistance regions or finite flat regions on the i–V curve. We show in the appendix, however, that negative resistance regions will appear if the a.c. source impedance becomes finite.

(2) Vertical current steps appear at rational voltages $V/V_\omega = n/m$. Since there is no reason to expect that trajectories starting at different points in the unit cell achieve a given rational slope

for the same value of A_0, it seems likely that there is a finite current step for *every* rational voltage, though of course the bulk of the steps will be too small to be observable.

(3) For very small A_J it can be shown that the steps at the multiple voltages $V/V_\omega = n$ have amplitude $2A_J\,|J_n(A_1)|$ where $J_n(x)$ is the Bessel function of order n. Thus as the excitation is raised from a low value these steps first have amplitudes proportional to $A_1^n/n!$, but when A_1 reaches n for a given step, the step amplitude will begin to oscillate, and to fall off again. Very similar behaviour is seen in the computation for comparatively large values of A_J. For very strong excitation, all steps, even the zero-order step, disappear.

(4) The steps at the submultiple voltages $V = nV_\omega/m$ are always smaller than the neighbouring multiple voltage steps, and their size falls off rapidly with m. By considering the separation in ϕ between equivalent trajectories we can produce plausible arguments to suggest how

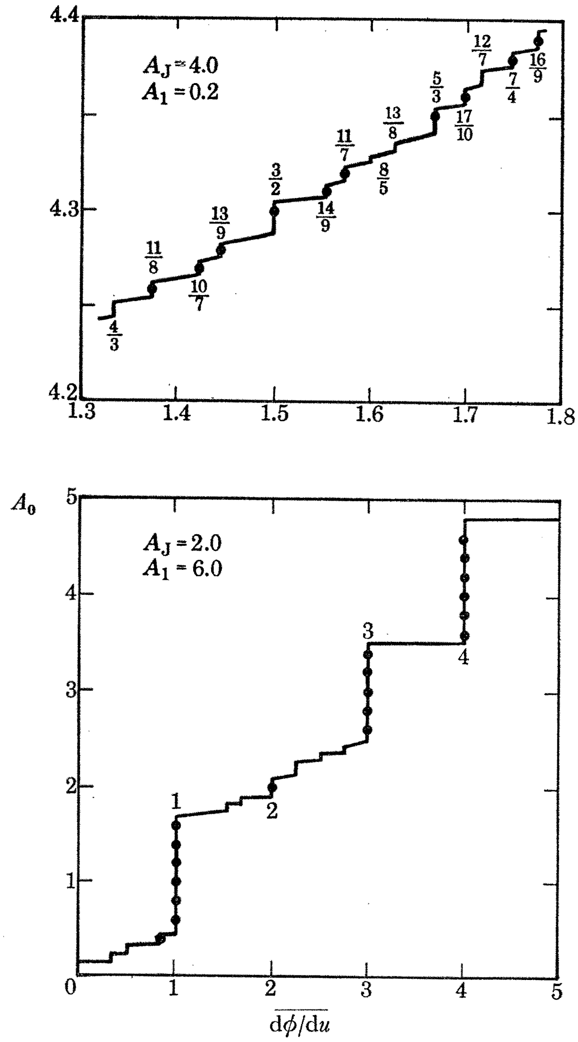

FIGURE 3. Two typical portions of computed i–V characteristics for small junctions with h.f. excitation. The upper diagram is for strong Josephson current and rather weak excitation; it shows rich submultiple structure between the first and second multiple steps. The lower diagram is for moderate Josephson current and rather strong excitation; it shows deep multiple steps with some submultiple structure. Notice that the excitation is so high that the step amplitudes have started to oscillate; the second step has almost disappeared at the value chosen. The points shown are cases in which the calculated voltage was within 10^{-3} of the fraction indicated.

large these steps should be. These arguments are, however, tedious and far from rigorous; rather than giving them in detail we prefer to state the following rules as conjectures:

For finite excitation we suppose that the characteristic consists entirely of steps, and that the step height falls off with m slightly faster than m^{-2}; steps close to a strong step of lower order tend to be further suppressed, however. As the excitation increases, we suppose that each step increases in size until it begins to be overlaid by a strong step of lower order, after which it shrinks. Beyond the point discussed in rule (3) above, at which the harmonic steps begin to decrease in size, we suppose that this process is reversed until at very strong excitation the characteristic consists entirely of limitingly small steps.

(5) In certain limits more definite conclusions about the submultiple steps can be drawn. For small Josephson current ($A_J \ll A_1$) the submultiple steps can be shown to be an order of magnitude smaller than the multiple steps. For small excitation ($A_1 \ll A_J$), on the other hand, the submultiple steps below the fundamental ($V = V_\omega/m$) are of first order in A_n, and all other steps are of higher order. The submultiple step amplitudes in this limit are complicated functions of A_J, but fall off roughly as m^{-3} with the order of the step. It is clear that the high-order submultiple steps are never really easy to see.

The above results were obtained for infinite a.c. source impedance. The effect of a finite source impedance, which may prove to be relevant to tunnel junctions at low frequencies, is discussed in the appendix. The effect of noise on the solutions of equation (12) is discussed in §6. The comparison of theory and experiment is deferred to the next section.

5. Large junctions

(i) *Basic equation and boundary conditions*

In the large junction we can no longer assume that the electric and magnetic fields and the phase difference ϕ are the same at all points in the junction. If we use Ampère's rule $\partial H/\partial x = j$ we find from equations (4), (6) and (7) that

$$(\hbar/2ed\mu_0)\,\partial^2\phi/\partial x^2 = j_J \sin\phi + \sigma V,$$

or

$$\lambda_J^2\,\partial^2\phi/\partial x^2 = \sin\phi + (1/\omega_J)\,\partial\phi/\partial t, \tag{14}$$

where ω_J is the frequency associated with j_J, given by $\hbar\omega_J = 2ej_J/\sigma$. In the same reduced units as were used in §4, this becomes

$$\partial^2\phi/\partial X^2 = \sin\phi + (1/A_J)\,\partial\phi/\partial u, \tag{15}$$

where X is x/λ_J.

The effect of the external sources is to determine the boundary conditions at the edge of the film, as in any situation where conduction tends to be concentrated in a skin region, and we should strictly speaking talk of sources of electric or magnetic field at the edge of the junction rather than voltage or current sources. Saying that the junction has a definite voltage 'across it' means that the field at the edge, and hence $\partial\phi/\partial t$ at the edge, takes a definite value. Similarly, specifying the current carried by the junction and the field in which it is placed is equivalent to specifying the magnetic field and hence $\partial\phi/\partial x$ at the edges. For a symmetrical junction in zero applied magnetic field the magnetic fields due to the current will be equal and opposite at the two edges, and using Ampère's rule shows that $2H_{\text{edge}} = Wj$ where j is the current injected expressed as a mean current density, and W is the width of the junction. Using (7) we have in reduced units

$$(\partial\phi/\partial X)_{\text{edge}} = \tfrac{1}{2}W\lambda_J^{-1}(j/j_J). \tag{16}$$

Since $\phi(x)$ is symmetric under these conditions we also have

$$(\partial\phi/\partial x)_{\text{centre}} = 0 \tag{17}$$

at the centre of the junction. In a junction in an applied field carrying zero current, on the other hand, $\phi(x)$ is antisymmetric, and $(\partial\phi/\partial X)$ is equal to $H/(\lambda_J j_J)$, while $\phi = 0$ at the centre of the junction. One should note in passing that it is not particularly easy to make perfectly symmetrical junctions and it is known that changes in the lead configurations or the current balance between paired leads can produce dramatic changes in the i–V characteristics, particularly when the a.c. and d.c. fields have different symmetries (Clarke, to be published). We also notice that for small junctions we can use the two conditions (16) and (17) to find an approximate expression for $\partial^2\phi/\partial X^2$:

$$\partial^2\phi/\partial X^2 \simeq 2\lambda_J W^{-1}[(\partial\phi/\partial X)_{\text{edge}} - (\partial\phi/\partial X)_{\text{centre}}] = A/A_J. \tag{18}$$

Using (15) we recover the small junction equation (12).

(ii) *Time independent solutions*

When ϕ is independent of time equation (14) becomes

$$\lambda_J^2\, \partial^2\phi/\partial x^2 = \sin\phi. \tag{19}$$

This equation also applies to tunnel junctions and has been thoroughly investigated (Ferrell & Prange 1963; Josephson 1965; Lebwohl & Stephen 1967; Owen & Scalapino 1967). Some of the solutions are illustrated in figure 4, by means of a mechanical analogue which we shall describe below. For small surface fields there is a solution in which the supercurrent flows near the edges of the junction, and the field is screened from the interior with penetration depth λ_J. If H at the surface exceeds $2\lambda_J j_J$, the phase at the surface will exceed π, the solution will become unstable, and a series of *flux-lines* will enter the junction. An isolated stationary flux-line is the solution

$$\cos\tfrac{1}{2}\phi = \tanh[(x-x_0)/\lambda_J]. \tag{20}$$

It is a region in which ϕ changes from $2\pi n$ to $2\pi(n\pm 1)$ and it contains exactly one quantum of flux, $\Phi_0 = h/2e$. If the specimen is strongly magnetized there will be a series of equally spaced flux lines in the interior (figure 4). Each flux line still contains flux Φ_0, but the flux density is now more uniform. The form of the solution is fixed by fixing the flux density, and the solution at a given flux density is metastable for external fields in the range

$$(\partial\phi/\partial x)_{\min} < H_S/(\lambda_J^2 j_J) < (\partial\phi/\partial x)_{\max}, \tag{21}$$

where $(\partial\phi/\partial x)_{\max}$ and $(\partial\phi/\partial x)_{\min}$ are the maximum and minimum gradients for the solution concerned. The external field for thermodynamic equilibrium lies within this range. This situation is sometimes described by saying that the system has a 'surface barrier' against flux motion. This surface barrier leads to hysteresis in magnetization experiments.

(iii) *Flux-flow*

The static solutions for the large junction show that the critical current corresponds to the point where flux lines begin to enter the junction. The special interest of equation (15) is that it is an explicit dynamic equation describing the flux-flow régime which is both realistic and soluble, something which has proved hard to find in the much more complex situation of type II superconductors. As a help in understanding its solutions it will be useful to bear in mind some general ideas concerning flux flow. What we have called a flux line is really a region in

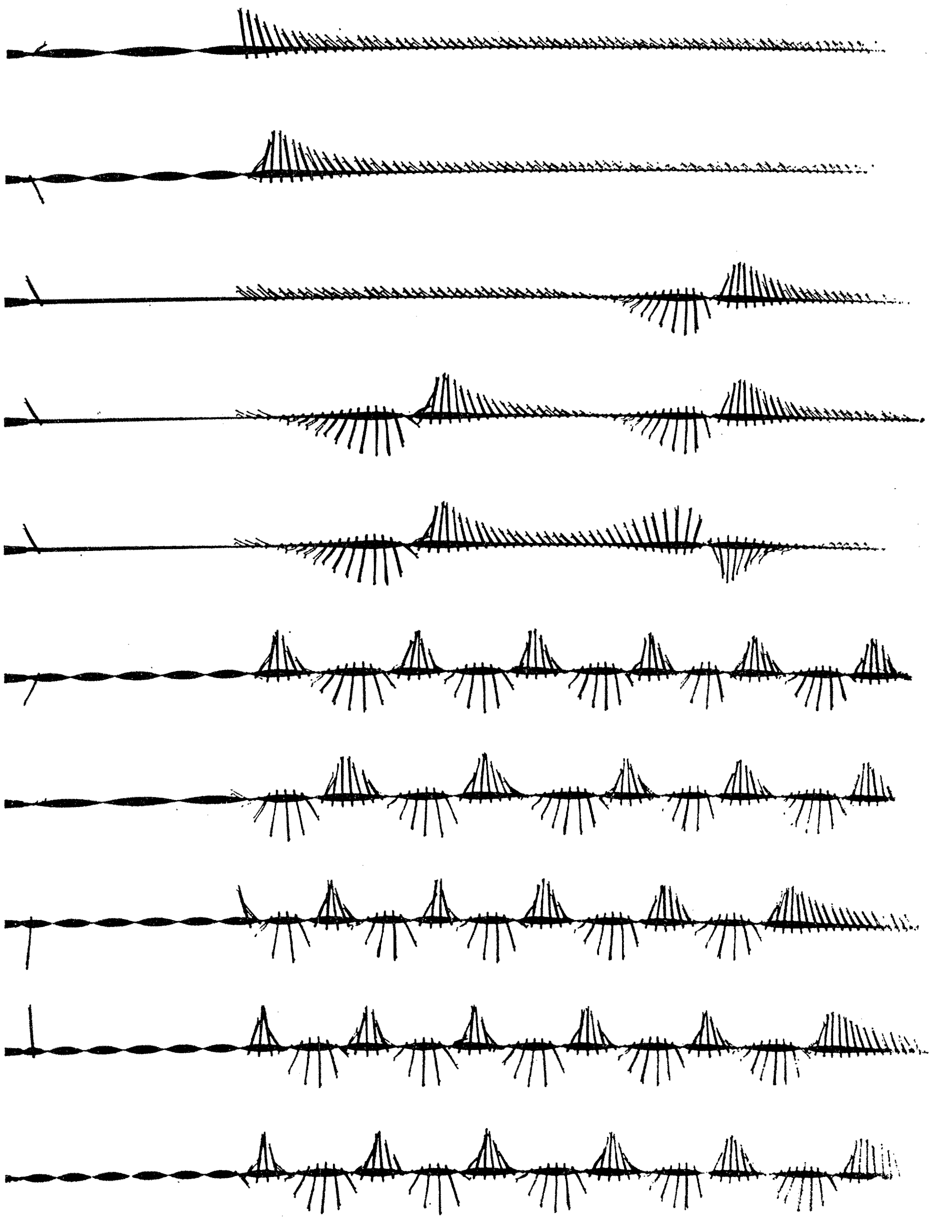

FIGURE 4. Photographs of the mechanical analogue under water. The point of view is almost directly above and slightly to the left. The twist in the elastic to the left represents the field applied to the edge of the junction. The centre of the junction is on the right. Cases 1 to 7 are static, but 8, 9 and 10 show movement. (1) A moderate applied field is screened from the centre of the junction by a surface supercurrent. (2) The critical field has been reached. If this field is exceeded, the system becomes unstable and a flux line enters the junction; in (3) the surface field has been reduced to zero, and the flux line is stationary deep inside the junction. (4) A second flux line of the same sign has been inserted. (5) Two flux lines of opposite sign. If they were a little closer their mutual attraction would pull them together, when they would annihilate each other. (6) Under a large applied field, many flux lines have entered the junction (the model is clamped on the right, so as to simulate the static-field boundary condition). If the field is reduced, the system does not become unstable until a much lower applied field is reached (7). This is the hysteresis effect associated with the surface barrier. Cases 8, 9 and 10 correspond to flux flow in the current-carrying state: the lines are moving from left to right. The density is highest on the left. Notice the flux line nearest the centre of the junction moving rapidly towards annihilation. (In cases 8, 9 and 10 the model is free to rotate on the right, so as to simulate the boundary condition for the current-carrying state.)

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which the phase change across the junction changes by 2π. It follows from (6) that the mean voltage across the junction at any point is $\nu\Phi_0$ where ν is the number of flux lines passing per second and Φ_0 is the flux quantum $h/2e$; this is true however closely the lines are packed, and shows that there is a definite flux-flow frequency associated with any voltage. The form of the equation for the voltage suggests that it is wholly magnetic in origin. This is a useful idea so long as the lines are well spaced. When they are close together the magnetic field becomes uniform, the mean supercurrent becomes zero and the junction is effectively normal. In this limit the electric field is just like the field in any normal wire carrying a current. To think of it as generated by the 'flux motion' is not especially helpful.

The separation between flux lines is determined by the driving forces acting on them and by the damping of their motion. As in type II superconductors flux lines of the same sign repel one another, and their motion is damped by losses generated by their internal electric fields. In order to produce flux flow it is necessary to set up a gradient of flux-line density which will be highest for large damping. What happens when the critical current is exceeded depends on the boundary conditions. In an external static magnetic field flux lines of the *same* sign enter from the edges and flow towards the centre until a static solution stable according to equation (21) is reached. But if the junction is carrying a d.c. current, flux lines of *opposite* sign enter the junction from the edges and flow towards the centre where they annihilate each other. In this case a dynamic equilibrium of permanent flux flow is established. As discussed above, there will be a d.c. voltage $\nu\Phi_0$, the same for all points in the junction.

We have not found any analytical solutions for equation (15) in the flux-flow régime. It is therefore particularly useful to note that the same equation is obeyed by a dense line of pendulums attached to a torsion fibre and immersed in a very viscous liquid. The angle of the pendulum to the downward vertical is ϕ, its angular velocity corresponds to the voltage across the junction, its projection $\sin\phi$ on the horizontal corresponds to the Josephson current density, the twist $\partial\phi/\partial x$ represents the magnetic field, and its derivative $\partial^2\phi/\partial x^2$ represents the total current density.† We have constructed such a model. Photographs of its static and dynamic behaviour are shown in figure 4. The flux-line density gradient needed to produce flux flow and the process of flux-line annihilation at the centre of the junction should be noted.

(iv) *The unexcited i–V characteristic*

We are particularly interested in the low-frequency i–V characteristics of large junctions in the flux-flow régime. In order to see what our theory predicts, we have solved (15) numerically for junctions of various widths under constant current conditions; the resulting i–V characteristics are shown in figure 5. The most interesting feature of the results is that for values of W/λ_J greater than about 4 some d.c. supercurrent persists up to large voltages; for large thicknesses the persistent part is about half the critical current. This effect can be understood as follows. In the flux-flow régime the supercurrent is an oscillating function of position which must be integrated to find the total current. At the centre of the junction the flux lines are

† More precisely the balance of couples acting on any pendulum is given by

$$SG\,\partial^2\phi/\partial x^2 = mgl\sin\phi + \rho\,\partial\phi/\partial t, \tag{22}$$

where S is the pendulum separation, G is the torsion constant, $mgl\sin\phi$ is the gravitational couple, and ρ is a viscous damping factor. Inertia is ignored. The corresponding analogue for Josephson junctions was noted independently by Scott & Johnson (1969). It has an inertia term corresponding to the self capacity (not present for SNS junctions), but no damping term, and therefore does not describe flux flow very realistically.

widely separated and moving fast. In these circumstances the periodic annihilation of equal and opposite flux-lines is a rapid process which occupies a rather small fraction of the total time; most of the time we may picture the flux-line nearest the centre as an almost isolated complete flux-line moving rapidly inwards. If we take such a situation and integrate the current from the centre of the junction outwards, the integral will oscillate once as we pass each flux line. However, as we near the edge of the junction the flux-line spacing decreases; for a wide junction carrying a large current it will become very small. This has the effect of decreasing the amplitude of the oscillating integral to a very small value. As often happens when an integral oscillates with decreasing amplitude, the limiting value is equal to half the initial oscillation. The initial oscillation represents the current carried by one side of a free flux line, which is the same as the critical current. Thus the persistent supercurrent is about half the critical current. This residue in fact persists until the current density is so high that the final flux line is itself appreciably compressed, which occurs at very high voltages for large junctions.

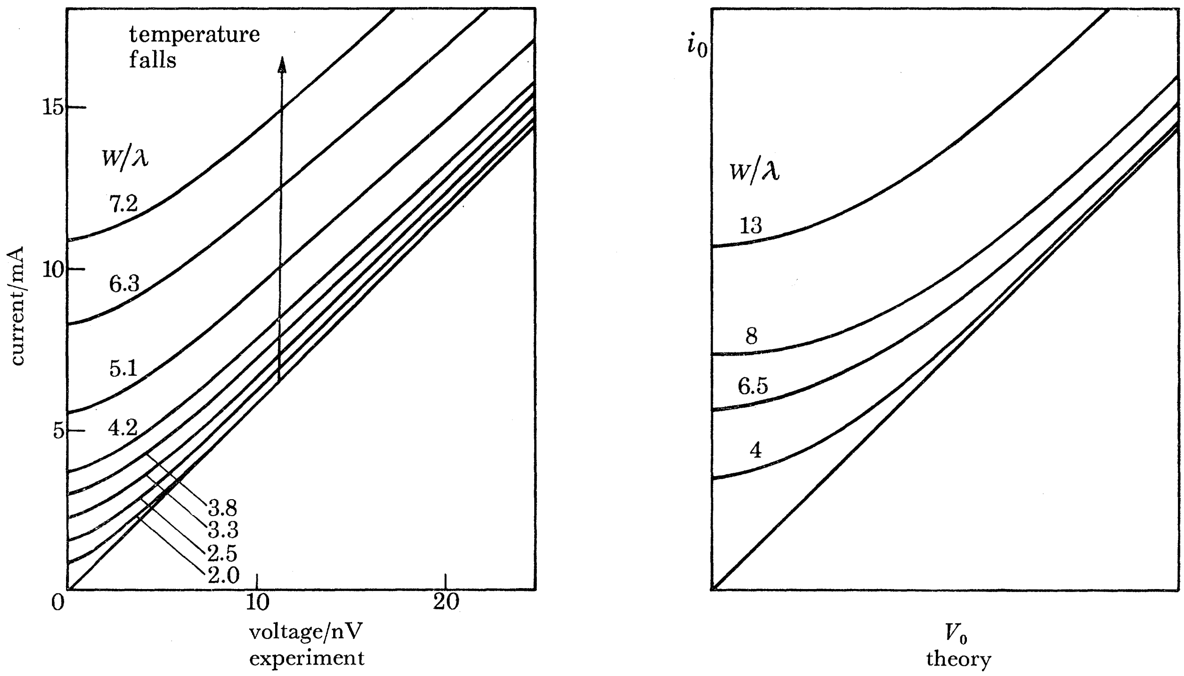

FIGURE 5. Experimental and theoretical d.c. current–voltage characteristics for a wide junction at different temperatures. The experimental curves refer to a square junction of side W = 0.2 mm; the copper layer was 1110 nm thick. The characteristic width λ_J decreases as the temperature falls. The theoretical curves refer to a one-dimensional junction of the same width and area.

The comparison of these predictions with experiment is confused by the fact that the only experimental data available are for square rather than 'long' junctions. We show in figure 5 some experimental results for a single square junction of fixed width W whose reduced width W/λ_J was varied by varying the temperature (Clarke, to be published). The labels W/λ_J were calculated by estimating λ_J from experiments on the self-field limiting of similar junctions of a range of sizes (Clarke 1969). The accompanying theoretical curves refer to a section of a 'long' junction of the same area, width and properties. For small junctions the critical currents for both sets of curves agree automatically because of the way the experimental curves were labelled, and the form of the i–V characteristic is in fair agreement with the small junction theory; in particular, there is no persistent supercurrent at large voltages. For larger junctions two effects are apparent. The first is that the experimental critical currents are larger than those predicted, though not greatly so. Since the square junction has twice the perimeter of the theoretical one, one might at first sight expect the critical current to be double the theoretical value for large junctions. However, it is clear that in a square junction the flux will begin to penetrate first at the corners, and it is not surprising that the critical current is substantially reduced because of this effect. The second notable effect is that the persistent supercurrent appears at lower values of W/λ_J and is larger than the 'long junction' model predicts. This disagreement is not surprising. The total supercurrent must be found by integrating over the area of the junction and until the pattern of flux flow in a square junction is understood it is not clear whether this integral should have a persistent value at high currents or not. The properties of square junctions are not easy to calculate theoretically, and the best hope of comparing theory with experiment properly seems to be to find ways of making effectively 'long' or circular experimental junctions.

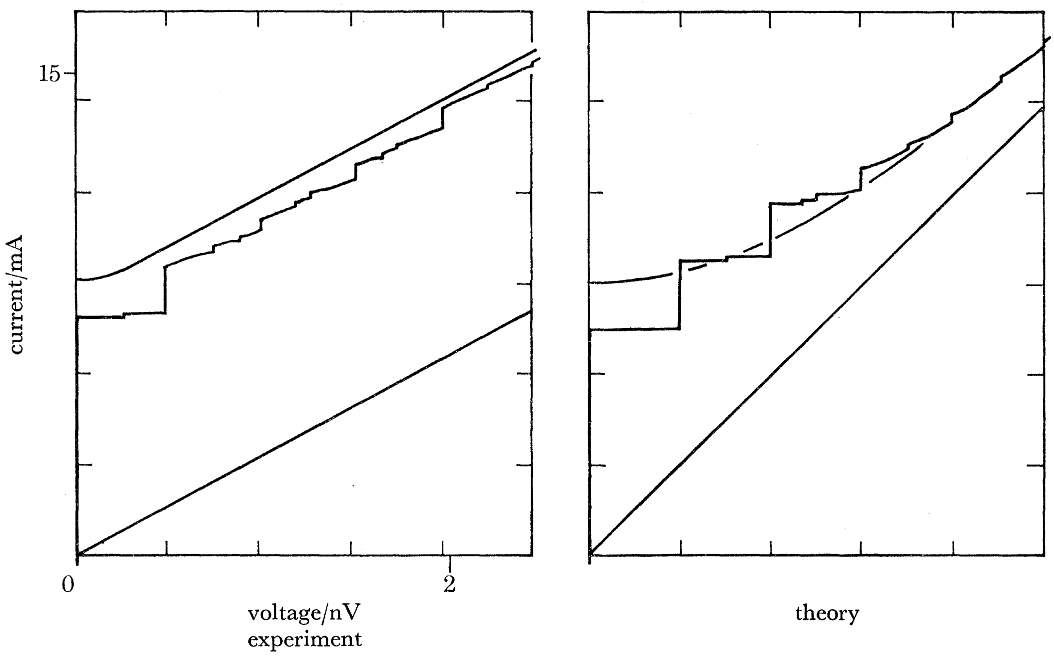

FIGURE 6. Experimental and theoretical d.c. current–voltage characteristics for a wide junction with and without h.f. excitation. The experimental results refer to a square junction of side $W = 0.2$ mm for which λ_J was 0.031 mm. The exciting frequency was about 250 kHz. The theoretical results refer to a one-dimensional junction of the same width. Its area was chosen so as to make the critical currents equal, which in this case gives the theoretical junction a substantially smaller resistance than the real junction.

(v) *The excited i–V characteristics*

The only reliable data on the i–V characteristics of excited junctions have been obtained on large, square junctions (Clarke, to be published), so that no direct comparison with experiment is yet possible for the small junction theory of §4. We have therefore extended the numerical solutions of (15) to the case where an oscillating current is superimposed on the steady current injected. This proved to be a time-consuming machine computation, and we have only carried through the calculation with any thoroughness for one set of values of A_J, A_1 and W/λ_J chosen to correspond roughly with the experimental results shown in figure 6. (Note that the current scales have been chosen to make the critical currents comparable, which makes the resistances appear rather different.) This calculation is satisfactory insofar as the sizes of the multiple and submultiple steps appearing are of very much the right order (and experience suggests that this agreement could have been improved by reducing the value of A_1 in the computation). But one must note that whereas in the experiments the excited characteristics was always below the unexcited characteristic, in the calculation the excitation appears to *enhance* the supercurrent at certain points. In this connexion it is interesting to recall the very strong enhancement of supercurrent by microwaves in Dayem bridges at certain frequencies (Dayem & Wiegand 1967). The depression of current in the present experiments may be associated with the special properties of square junctions and need not be considered as showing a failure of our model.

It is interesting to consider how far the exciting fields penetrate into the junction. Explicit calculation shows that a strong exciting field leads to large perturbations of the flux flow pattern within about λ_J of the surface, but that further from the surface the orderly pattern is very little disturbed. Thus the d.c. electric field and the a.c. electric and magnetic fields at the multiples of the Josephson frequency, which are carried by the flux lines, penetrate right through the junction, while the field components at the excitation frequency are limited to the edge region.

The distance over which the exciting field penetrates is in fact approximately what one would expect from ordinary skin effect theory. In a bulk superconductor the h.f. skin effect is close to the smaller of the penetration depth and the classical skin depth, whose analogues in the SNS junction are easily found to be λ_J and $\lambda_J A_J^{\frac{1}{2}}$. The calculations mentioned were done for values of A_J near to unity; for smaller values of A_J it seems likely that the exciting field penetrates an even smaller distance than λ_J.

6. Noise

As we pointed out in §2, the current steps in the excited i–V characteristics of SNS junctions are exceedingly steep, much steeper than those for tunnel junctions. The voltages near the top and bottom of the steps differ by less than one part in 10^8 (Clarke 1968). Since measurements of these voltages can in principle be used to determine the ratio e/h to a high accuracy (Parker *et al.* 1969), it is particularly important to understand why these steps are so precise and why they are not appreciably broadened by noise. The analysis that follows is based on the small junction limit (see §4).

Let us consider the solutions of equation (12) corresponding to points on the step of voltage nV_ω/m, that is, trajectories which are locked on to the lattice in ϕ, u space and are exactly periodic over m periods of u and n periods of ϕ. Starting at any point in the ϕ, u plane there will be some value of A_0 which will generate such a trajectory. Let us plot out all such trajectories and label them with their A_0 values, which we now call A_p, to show that they refer to the *periodic* trajectories for the step of interest (figure 2). The value of A_p is a function of position in the plane whose contours are the periodic trajectories. In the figure, A_p rises as we pass from AA through BB to CC (shaded region) and then falls again until the next shaded region is reached. We can now ask what happens if the d.c. current (or A_0) corresponds to a trajectory such as BB, but the system starts at some point not on BB. If this point lies between BB and CC then A_0 must be less than the local value of A_p. Equation (12) shows that the system will move along a trajectory of smaller slope than the local periodic trajectory, which will bring it closer to BB. This process will continue until the system converges on to the periodic trajectory BB for which $A_p = A_0$. If the system starts between AA and BB it will converge *upwards* on to BB by a similar process. This makes clear that BB is an *actively stabilized trajectory*. Small fluctuations in phase due to noise are rapidly corrected and do not affect the long-term mean slope of the trajectory (or d.c. voltage).

This does not mean that noise has no effect on the voltage, however. Between CC and AA in the unshaded region is another periodic trajectory with the same value of A_p as BB had. This trajectory is clearly *unstable*, and if the system starts near it, it will diverge from it and converge on to BB or the nearest equivalent trajectory in one of the stable regions, which are shaded in the figure. If a noise fluctuation is large enough to carry the system from a stable trajectory beyond the nearest corresponding unstable trajectory, the system will probably converge into a different stable band from its starting trajectory. If this happens frequently the mean slope $\overline{\partial\phi/\partial u}$ will be affected, the d.c. voltage will drift, and the step will show 'noise rounding'. This jumping from one stable trajectory to another is a hopping process activated by noise not unlike the diffusion of atoms in a solid. This noise rounding will of course mean that the Josephson relation $n\hbar\omega_0 = 2\Delta\mu$ is no longer strictly valid at all points on the step. This conclusion does not violate any fundamental principle; it is a simple consequence of the phase slippage between the a.c. supercurrent and the applied a.c. signal in the presence of noise. We expect the rounding to be symmetrical about the midpoint of the step, so that values of the voltage measured there can be accepted as unbiassed in making measurements of the ration e/h.

We now attempt to make these ideas quantitative. We first ask how the local value of $A_p(\phi, u)$ varies with ϕ for arbitrary u. It is an oscillating function with m maxima and m minima in each unit cell, and these maxima and minima are all equal and independent of u. A convenient approximation is to assume that it is sinusoidal. In fact, we write

$$A_p = A_c + A_s \sin[m(\phi - \phi_c)], \tag{23}$$

where $\phi_c(u)$ is the central stable trajectory which corresponds to the midpoint of the current step and A_c is the corresponding value of A_p. Clearly A_{max} is $A_c + A_s$ and A_{min} is $A_c - A_s$, so that $2A_s$ is the height of the step concerned. We notice that for fixed A_p this expression implies that $\phi - \phi_c$ is independent of u, that is, that all the periodic trajectories run parallel in the ϕ, u plane. This is clearly only a crude approximation for low-order steps (and also for high-order steps near to a low-order step), but it represents the essential physics sufficiently well for the moment. We shall show below that this approximation actually has no gross effect on the noise rounding calculated for the steps. If we now change our phase variable to $\theta_0 = \phi - \phi_c$, we find using (12) and (23) that

$$\begin{aligned} \mathrm{d}\theta_0/\mathrm{d}u &= \mathrm{d}\phi/\mathrm{d}u - \mathrm{d}\phi_c/\mathrm{d}u \\ &= \mathrm{d}\phi/\mathrm{d}u - \mathrm{d}\phi_p/\mathrm{d}u \\ &= A_0 - A_p \\ &= \Delta A_0 - A_s \sin(m\theta_0), \end{aligned} \tag{24}$$

where ΔA_0 is $A_0 - A_c$. This equation expresses the active phase-locking of the system in a quantitative way.

We now consider how noise will affect this equation of motion. For SNS junctions calculation shows and experiment partly confirms that the internal Johnson noise of the junction is much more important than either external noise or shot noise. With current sources the effect of internal noise is to add a noise voltage to the term σV in equation (11). Following this through we find that (24) is replaced by

$$\mathrm{d}\theta_0/\mathrm{d}u = \Delta A_0 - A_s \sin(m\theta_0) + A_N(u). \tag{25}$$

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The noise term has a uniform (white) power spectrum whose density may be determined by noting that the fluctuations of the total normal current have a spectral density $2kT/\pi R$ per unit range of angular frequency, where R is the resistance of the junction ($1/(\sigma S)$ for a junction of area S). Thus $A_N(\omega)$ has a density $8e^2RkT/(\pi\hbar^2\omega^2)$, and $A_N(u)$ a density $8E^2RkT/(\pi\hbar^2\omega)$. We may now interpret (25) by means of the analogue of a particle on a gently sloping washboard (sinusoidal potential) in a viscous liquid; if θ_0 is the distance variable and u the time, the first two terms on the right of (25) describe the terminal velocity at any point, while the last term describes the Brownian motion. It is convenient to treat this as a diffusive process, with a diffusion constant D that may be written down from the spectral density of velocity,

$$D = 2\pi e^2RkT/(\hbar^2\omega) = \tfrac{1}{2}\pi\overline{V_N^2}/V_\omega^2, \tag{26}$$

in which $\overline{V_N^2}$ is the mean square noise voltage in the frequency band 0 to ω. The problem of Brownian motion under these conditions has already been solved by Ambegaokar & Halperin (1969) in connexion with the closely related problem of the d.c. Josephson effect, and we may quote the result of their analysis as follows: a particle which without Brownian motion would have local terminal velocity $v_0[1-\beta\sin(m\theta_0)]$, when influenced by Brownian motion drifts at a mean rate given by

$$v = \tfrac{1}{2}mD\sinh(\pi q)\left[\int_0^{\frac{1}{2}\pi}\cosh(2qx)I_0(2q\beta\cos x)\,\mathrm{d}x\right]^{-1}, \tag{27}$$

where $q = v_0/(mD)$.† I_0 is the modified Bessel function. This expression holds both for $\beta < 1$, when the particle can move even without fluctuations to help it, and when $\beta > 1$ and fluctuations are needed to help it over the crests; it is the latter case which interests us.

To interpret the model solution we note that $v_0 = \Delta A_0$ and $\beta = A_s/\Delta A_0$, so that by use of (26)

$$q = 2V_\omega^2\Delta A_0(m\pi\overline{V_N^2})^{-1}, \quad q\beta = 2V_\omega^2A_s(m\pi\overline{V_N^2})^{-1}. \tag{28}$$

Further, v is the mean rate of phase slippage, which can be taken as a correction to $\overline{\mathrm{d}\phi/\mathrm{d}u}$ in a diagram like figure 2, causing the steps to be not quite vertical. The value of $\mathrm{d}v/\mathrm{d}(\Delta A_0)$ gives the differential resistance R' of the junction in terms of its d.c. resistance R, and we shall calculate this at the centre of the step, where $\Delta A_0 = 0$. Since the integral in (27) remains finite when q vanishes, we need take only the derivative of the numerator, to yield

$$\begin{aligned} R'(0)/R &= \tfrac{1}{2}\pi\left[\int_0^{\frac{1}{2}\pi}I_0(2q\beta\cos x)\,\mathrm{d}x\right]^{-1} \\ &= [I_0(q\beta)]^{-2}, \end{aligned} \tag{29}$$

by use of Neumann's formula (Erdélyi 1953). The argument of I_0 in (29) may be written using (28) as i_S/mi_N where i_N is a characteristic noise current equal to $2kT/\Phi_0$ and i_S is half the step height. The form of this result is very reasonable. The argument of the modified Bessel function (which is similar to the exponential function) reaches unity when $\frac{1}{2}i_S\Phi_0/m$ is equal to kT. The noise source has to inject a current i_S and the phase must drift by $\pi/2m$ if hopping to another trajectory is to occur, and the work done by the noise source in this process is of order $\frac{1}{4}i_S\,\Phi_0/m$. Equation (29) shows that phase drift due to trajectory hopping is exponentially unlikely for steps large enough to make this energy greater than kT. This general interpretation shows that the approximation involved in (24) is not serious. The true analogue for the trajectories is a

† In equation (9) of Ambegoakar & Halperin's letter, the lower bound of the last integral should be θ instead of 0. The double integrals in (9) have been simplified into the form (27) by manipulation of the domain of integration and change of variables.

washboard which periodically changes its shape somewhat, but has the same *average* period and the same depth as the sinusoidal washboard. It is clear on general grounds that the hopping probability will not be drastically modified by this refinement.

The value of i_N at a noise temperature of 4 K is rather less than 10^{-7} A. In a typical experiment on an SNS junction near 1 MHz, the major steps have heights of order 10^{-3} A, and one finds that $R'(0)/R$ should be about $\exp(-10^4)$ Our result implies that subharmonic steps of very high order should be visible if looked for carefully, though this may require the use of rather well-stabilized sources. Alternatively, it implies that steps should be visible in very weak junctions with critical currents in the region of 10^{-6} A, which would most easily be examined at very low frequencies of the order of 1 kHz.

7. Conclusions

We have shown that our model of the SNS junction, which is really quite crude, is capable of describing rather well the qualitative behaviour of the i–V characteristics of both large and small excited junctions, and there is reason to hope that even better agreement will be obtained for large junctions when proper account is taken of their special geometries. It is therefore important to ask what features of our model are most significant. It seems that three things are outstandingly important. The first is the assumption that the Josephson current is a *periodic* function of ϕ. Changing the form of this function or making it weakly dependent on voltage would probably not change our conclusions drastically so long as the function remained periodic. The second is the inclusion of a term describing *normal current flow*; again, the precise form may not be very important, but the existence of an alternative to supercurrent, and the presence of flux-flow damping have been essential to our arguments. The third is the careful consideration of *h.f. source impedance* which has an over-riding importance in determining the form of the i–V characteristic and its response to noise.

We would also like to emphasize, following Anderson (1967), that the a.c. effect is essentially a phase-locking mechanism. We have to bear in mind that at finite voltage the weak link has an internal frequency (the flux-flow frequency, or, in a small junction the a.c. supercurrent frequency). If this can become phase-locked to the applied frequency, power can be fed continuously from one system to the other, the direction of flow depending on which system would run faster, left to itself. Since the energy of the junction is drawn from the d.c. source, changes in the energy flow are reflected as changes in d.c. current, and a current step at the phase-locked voltage results. It is essentially the phase-locking mechanism which makes the d.c. voltage so insensitive to noise. This is brought out very clearly by the discussion of the small junction behaviour in §4. The fact that (12) is non-linear means that any deviation from the stable trajectory produced by noise is self-correcting. It is only when the noise can make the system jump to the next stable trajectory that noise begins to matter.

Although our calculation is performed in the classical limit, it may be helpful to point out that the multiple and submultiple steps in the i–V characteristic correspond to well-defined quantum processes in which n photons of radiation at the exciting frequency are absorbed or emitted, and m pairs of electrons cross the junction, giving the energy relation $n\hbar\omega = m2eV$. As usual in quantum mechanics, the multi-photon processes involving large n are only important for large photon densities (large A_1). So far as the m quantum number is concerned the junction behaves like an oscillator of frequency $2eV/\hbar$ with equally spaced levels. For a linear oscillator the selection rule limits the transitions to single steps up or down the ladder of levels ($m = 1$), but for nonlinear oscillators other transitions become possible and increase in strength with the nonlinearity (large A_J). These ideas correspond with the calculations of §4.

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Appendix. Arbitrary source impedance and Josephson junctions

For an arbitrary frequency-dependent complex source impedance it is difficult to suggest general methods of treating the equations. In some cases, however, it may be reasonable to treat the source impedance as resistive. In the small junction limit the effect is simply to introduce this impedance in parallel with the current source, and therefore in parallel also with the junction. In this case the relation between the d.c. current source i_0 and the d.c. voltage V will be exactly as calculated in §4 above except that the normal current conductance $1/R$ is replaced by $1/R+1/R_S$ where R_S is the source resistance. In comparing with experiment, however, one must remember that the measured current is not i_0, but the current entering the junction, i_0-V/R_S. The effect of these changes on the i–V characteristic is as follows: (i) The normal state characteristic is unchanged. (ii) The parameters, A_1, and A_S, decrease. In the limit of small R_S, the submultiple steps vanish, and the heights of the multiple steps tend to a constant limit. (iii) The subtracted term V/R_S leads to negative resistance regions on either side of the steps. In the limit of small R_S these become very deep, and the steps revert to the 'spikes' of the Josephson calculation.

It is also possible to carry through the calculation for different a.c. and d.c. source impedances. It turns out that the i–V characteristic is independent of the d.c. source impedance and equal to that calculated above with R_S set equal to the a.c. source impedance. In fact the only importance of the d.c. source impedance is in fixing the load-line, which determines whether the system shows hysteresis or not. We notice that according to this analysis the absence of visible negative resistance regions in SNS junctions implies that the a.c. source impedance is very high, while the absence of visible submultiple voltage structure in tunnel junctions implies that the a.c. source impedance is very low.

The fact that the junction resistance R and the source resistance R_S are in parallel also makes it easy to handle the noise calculation in this case. The noise is most conveniently treated as a current source with spectral density for i^2 of $2kT/(\pi R)$ in parallel with the resistance concerned. The internal noise is mainly Johnson noise of the normal current (shot noise is normally negligible), and the external noise can be represented as an equivalent noise temperature in the source resistance. It is easy to see that internal noise will dominate if T/R is greater than T_S/R_S. This is certainly so in the SNS junction where R is of order $10^{-7}\,\Omega$, while R_S must be at least $1\,\Omega$. In tunnel junctions, the reverse is usually the case. It is clear that the noise calculation of §6 can be extended to the case of arbitrary source impedance by replacing $1/R$ by $1/R+1/R_S$ and assigning it an appropriate noise temperature, which is easily found to be $(TR_S+T_S R)/(R+R_S)$.

We finally turn to the question of which parts of our analysis can be applied to tunnel junctions. The most important difference in this case is the presence of a substantial internal capacitance. In §§ 3 and 4 above, where we considered small junctions, we assumed that the effect of a large capacitance is to make the h.f. source a voltage source, and we found that this led to the appearance of current 'spikes' rather than 'steps'. Some authors who have considered the full equation of motion have not reached the same conclusion, and we here compare our work with theirs. If one writes down the current equations for a small junction of capacitance C fed from a source of impedance R_S one obtains the equation of motion (Lebwohl & Stephen 1967).

$$\frac{C\hbar}{2e}\ddot{\phi}+\frac{\hbar}{2cR_{\text{eff}}}\dot{\phi} = i_0 - i_J \sin\phi + i_1 \sin(\omega t) + i_N(t). \tag{30}$$

Here $1/R_{\text{eff}}$ is $1/R_S + 1/R$ as above, and $i_N(t)$ is the noise current. We expect that this second-order equation will show the same phase-locking phenomena as did equation (12), but with some important differences. Ambegaokar & Halperin (1969) have examined this equation for the case $i_0 = 0$ (no excitation) in order to analyse the noise-rounding of the zero-voltage Josephson 'spike'. They point out that the equation has as a mechanical interpretation the Brownian motion of a massive particle in a viscous fluid on a tilted washboard, and they solve this problem in the heavy damping limit (small R_{eff}). This leads them to what is essentially our (25) corresponding to a current step in the i–V characteristic, and not a spike. However, the heavy damping limit is only valid if $\omega\tau \ll 1$, where τ is CR_{eff}, the decay time of the condenser, and ω is a typical frequency of the solution, say $I_J R_{\text{eff}}/\Phi_0$. This is equivalent to saying that the impedance of the capacitance must be much greater than R_{eff} at the typical frequency. As we pointed out in § 3, this is not true in typical tunnel junctions. In fact one can see intuitively that for large C (which corresponds to large inertia) the mechanical analogue can have two solutions for a given angle of tilt. In one the particle is at rest in a valley of the washboard, while in the other the particle moves steadily down the washboard, using its inertia to punch through the potential hills. If C is large enough, its velocity will be virtually constant, and there will be no d.c. supercurrent. This is the ordinary tunnel characteristic solution at finite voltage in the current 'spike' case. The correct calculation of noise rounding in this case requires the full analysis of the Fokker–Planck equation which Ambegaokar & Halperin write down but do not solve. In the case $i_1 \neq 0$ one can show by iteration of (30) that a very similar equation will also apply for the finite voltage steps of the excited characteristic. In this connexion, we must draw attention to the fact that Stephen (1969 *b*) has attempted to solve this problem. After various approximations he also reaches an equation essentially identical with (25) above. What is surprising is that this calculation uses an a.c. *voltage* source, for which we would again expect to see a current 'spike' rather than a 'step'. It is therefore important to make clear the significant differences of approach. The first is that in Stephen's calculation an important role is played by the transmission-line mode of the junction nearest to the Josephson frequency; the calculation assumed that such a mode will not be far away and will contribute to the voltage across the junction. We, on the other hand, have ignored such modes because in SNS junctions they do not exist, while in typical small tunnel junctions even the lowest lying mode will be well above the Josephson frequency. Even in large junctions it is only at the special voltages for which the Josephson frequency is particularly close to such modes that they appreciably affect the behaviour; this is the phenomenon of 'Fiske steps' (Fiske 1964; Pippard 1964; Josephson 1965). However, the form of Stephen's result is not appreciably altered if the junction mode is suppressed, so one must look elsewhere for the disagreement. The second difference is in the treatment of source impedance. Beginning by using an a.c. voltage source Stephen extracts a current conservation equation (equation (27)) which is supposed to involve only slowly varying quantities. In this equation the external circuit is represented as a d.c. current source. This assumption is not at variance with the use of a voltage source at high frequency so long as the quantities in the equation are really slowly varying. However, it seems to us that this is actually only true in general in a very small region close to the critical current for the step concerned. In particular it seems to be untrue during the response to a noise fluctuation at most points on the step, and also once the critical current is exceeded. The presence of comparatively rapidly varying currents in an equation involving a current source would imply that the high-frequency source impedance was large, contrary to the original model. It seems to us that a high a.c. source impedance has been accidentally introduced into the calculation at this point.

The internal capacitance is equally important in large tunnel junctions, and the barrier equation in this case is (Lebwohl & Stephen 1967)

$$\frac{\partial^2\phi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=\frac{1}{\lambda_J^2}\sin\phi+\frac{1}{\omega_J\lambda_J^2}\frac{\partial\phi}{\partial t}. \tag{31}$$

The corresponding equation without the final damping term has been investigated by Josephson (1966), Stephen (1969*a*) and Scott & Johnson (1969). The flux-flow solutions of (31) and the process of flux annihilation are complicated and are not completely understood.

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